Linear stability analysis of bifurcations with a spatially periodic, fluctuating control parameter

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Multiplicative noise in spatially extended systems produces different effects depending upon whether the noise is spatially homogeneous or spatially varying. Whereas in previous work a stochastic distribution was treated, here we consider the spatially periodic case, which is more amenable to an experimental approach, in particular in the electrically driven instabilities of nematic liquid crystals. We shall principally be interested in the threshold for the onset of symmetry breaking instabilities controlled by bifurcations in several stochastic partial differential equations. For the Ginzburg-Landau and Swift-Hohenberg equations we calculate the behavior of the threshold for all moments to second order in the noise strength, allowing one to reconstruct the full probability distribution. For a system of two coupled equations which mimics electroconvection in nematic liquid crystals (the ''one-dimensional model''), we calculate the first two moments up to second order and estimate the threshold for convection. The general conclusion of our work is that spatially periodic noise induces a reduction in the threshold similar to the stochastically distributed case. We propose that this reduction be independent of the periodicity of the noise to first order in the noise strength, the dependence on period appearing only at second order. This is in contrast to spatially homogeneous noise where threshold shifts may be entirely absent. [S1063-651X(97)15305-3]

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I. INTRODUCTION

The way in which fluctuations of the control parameter influence spatially extended systems exhibiting symmetry breaking instabilities, such as the transition to electrohydrodynamic convection (EHC) or the electrically driven Fréedericksz transition in liquid crystals, has recently been the focus of some interest both experimentally [1–7] and theoretically [8–31]. The fluctuations of the control parameter, in the case of liquid crystals the applied voltage, introduce a noise term in the stochastic partial differential equations (SPDEs) which is (to lowest order) proportional to the stochastic field(s) and is hence termed multiplicative (one might also say parametric). Such systems can be described by a single stochastic differential equation of first order in time or a set of coupled SPDEs. The case of several SPDEs is important since a set of SPDEs near the threshold of the instability cannot be reduced directly to a single stochastic amplitude equation if multiplicative noise is present. This is in contrast to the case of (weak) additive noise where such a reduction is possible [32–43]. Measurements of the effects of noise in experimental systems without (intentionally) introducing external noise have been done in a number of cases [44–52], some of which revealed a noise strength compatible with thermal noise [48–52].

Since multiplicative noise does not destroy the basic (primary) solution [53] one still has a sharp bifurcation (for a stationary noise process) and the threshold can be calculated by linearizing the SPDEs around the basic solution. (This can be shown under rather general conditions [30,31].) Thus one can uniquely define a threshold $a(\epsilon)_{\mathrm{threshold}}$ of the deterministic (time-averaged) control parameter a in the presence of multiplicative noise of strength ϵ, below which small nontrivial initial conditions decay to zero and above which they diverge exponentially with probability 1 (''almost certainly''). One may also define a threshold a_n for the nth moment, which may, in principle, be smaller than $a_{\mathrm{threshold}}$.

The effect of spatially constant multiplicative noise, which occurs when the control parameter has only temporal fluctuations, has mainly been considered [1–7,15–20]. Then the linearized equations are amenable to the usual Fourier decomposition in the extended spatial directions, where one assumes translational invariance. Then one is often left with ordinary SDEs in time, just as for restricted (zero-dimensional) systems. In the case of a single ordinary SDE, which has been investigated intensely in the past [8–15], one finds that the a_n are lowered by an amount of order $n\epsilon$ whereas the threshold $a_{\mathrm{threshold}}$ is not at all affected by the noise. This peculiar behavior results from the fact that the distribution function of the linearized problem has long tails. In the nonlinear problem all thresholds coincide with $a_{\mathrm{threshold}}$. Unfortunately it has not been possible to treat the nonlinear problem for more complicated cases. The case of two coupled SPDEs has been considered in the context of EHC [16–20]. Here one expects (and finds) a threshold shift of order ϵ.

The case of a spatially stochastically varying multiplicative noise term in a single SPDE (or a discrete version on a lattice) has also been recently studied, and found to produce interesting effects [21–31,54,55].

An interesting aspect, first found in simulations [21], was a shift of order ϵ of the threshold towards smaller values of the control parameter without introducing significant fluctuations into the system. The authors of [21] also noted that, within numerical accuracy, this shift was equal to that calcu-

*Present address: Los Alamos National Laboratory, Los Alamos, NM 87545 lated for the second moment (or correlation function). A rather general approximate theoretical approach has been developed to describe this threshold shift [29–31]. The method employed makes an ansatz for the form of the probability distribution of the stochastic field, which can be justified for a number of cases, and allows one to calculate the shifted threshold to $O(\epsilon)$, and often to $O(\epsilon^2)$, from the knowledge of the long-term behavior of only the first two moments of the stochastic field. Here the threshold of the second moment differs from that of the first moment (and of the actual threshold) by corrections that are typically of order ϵ^2 and which could not have been detected in the simulations of Ref. [21]. By letting the correlation length of the noise become small—or equivalently the correlation length of the deterministic part large—the prefactor in the correction term can be made arbitarily small, and then the system presumably behaves fully deterministically (in the absence of additive noise). In this limit the results of the analysis coincide with those obtained from a type of mean-field approximation of the system including weak additive noise [22] (in the scaling used there the weak-noise limit corresponds to the strong-coupling limit).

In this paper we present the results of a study of the effect on the theshold for multiplicative noise which is spatially deterministically modulated and temporally fluctuating on three SPDEs in one spatial dimension. More specifically we chose the noise term in our SPDE to be the product of white noise $\xi(t)$ [i.e., $\langle \xi(t)\xi(t')\rangle = \delta(t-t')$, where $\langle \cdots \rangle$ denotes an ensemble average] and a periodic function $f(x)$. In electrically driven instabilities in liquid crystals this can be achieved by employing structured electrodes, as done previously in other contexts [56,57]. We first consider the (real) Ginzburg Landau equation (GLE), which involves only one stochastic field and should be applicable to simple spatially homogeneous transitions like the usual Fréedericksz transition (see, e.g., [58]). Then we treat the Swift-Hohenberg equation (SHE), which also involves only a single stochastic field $\Psi(x,t)$, but has the instability at nonzero wave number, and then later the case of two coupled stochastic fields that describe one-dimensional EHC; see [59,60,58,61,17–20]. Since we consider only the threshold behavior, the equations studied in this paper are all linear (see Appendix A in [30]). In the case of the GLE (in Sec. II) and the SHE (in Sec. III) we shall illustrate that we can calculate perturbatively in the noise strength the long term behavior of all integer moments of the stochastic field $\Psi(x,t)$ and hence reconstruct the full probability distribution. To second order in noise strength this will be seen to be precisely of the form suggested in the ansatz due to Becker and Kramer [29–31]. In Sec. IV we give results for the first two moments to second order in noise strength for the more complicated case of the two coupled SPDEs of the 1D EHC theory. These can be seen to take a similar form to those of the simpler single SPDEs in Secs. II and III. In the conclusion (Sec. V) we make connection with further work and discuss the mechanism underlying the threshold shift.

II. THE GINZBURG-LANDAU EQUATION

In this section we shall consider the GLE in one dimension with multiplicative noise, which is the product of white noise in time and a deterministic periodic function in space, i.e., the equation takes the form

$$\frac{\partial}{\partial t}\Psi(x,t) = \left(a + \frac{\partial^2}{\partial x^2}\right)\Psi(x,t) + \sqrt{\epsilon} f(x)\Psi(x,t)\xi_t + c\Psi^3(x,t), \tag{1}$$

where the above is to be understood in the physically relevant Stratonovitch interpretation (midpoint discretization, see [62]). The deterministic control parameter here is a, ϵ varies the strength of the noise, $f(x)$ is a periodic function with period $2\pi/Q$, and c is a constant. Above we have given the full nonlinear GLE. To determine the onset of the instability we shall only need to concern ourselves with the linearized equation, i.e., $c=0$.

Let us briefly consider the deterministic equation ($\epsilon=0$). This has plane wave solutions e^{ikx} with corresponding eigenvalues $a-k^2$. The critical mode, where the eigenvalue is maximal, and which will become unstable first, is therefore that with zero wave vector, $k=0$. When $a<a_{\text{threshold}}=0$, all eigenvalues are negative and there exists only one stable stationary solution, $\Psi(x,t)=0$. For $a>a_{\text{threshold}}$ positive eigenvalues arise and although this stationary solution still exists, it is unstable. The new stable stationary solution cannot be determined from the linearized equation. To do this one would need to include the dominant nonlinearities present in the full SPDE, which control the bifurcations.

Having noted that in the absence of noise the threshold is at $a_{\text{threshold}}=0$, we shall now consider the case $\epsilon \neq 0$. To obtain information about the probability distribution of the field $\Psi(x,t)$ we shall calculate the moments of the field, i.e., $\langle \Psi(x_1,t)\Psi(x_2,t)\cdots\Psi(x_n,t)\rangle$. To do this we shall first transform from the Stratonovitch to the Ito interpretation of SPDEs to allow us to use the results of Ito calculus [62]. Doing this one obtains the Ito SDE,

$$\frac{\partial}{\partial t}\Psi(x,t) = \left(a + \frac{\partial^2}{\partial x^2} + \frac{\epsilon}{2}f^2(x)\right)\Psi(x,t) + \sqrt{\epsilon} f(x)\Psi(x,t)\xi_t. \tag{2}$$

Using the results of Ito calculus [62] one can now write down equations for the integer moments of the field. For the nth moment one sees that

$$\frac{\partial}{\partial t}\langle \Psi(x_1,t)\Psi(x_2,t)\cdots\Psi(x_n,t)\rangle = \left[\sum_{i=1}^{n}\left(a + \frac{\partial^2}{\partial x_i^2}\right) + \frac{\epsilon}{2}\left(\sum_{i=1}^{n} f(x_i)\right)^2\right] \times \langle \Psi(x_1,t)\Psi(x_2,t)\cdots\Psi(x_n,t)\rangle. \tag{3}$$

Readers unfamiliar with Ito calculus can obtain Eq. (3) directly from Eq. (1) by writing down the equation satisfied by $(\partial/\partial t)\Pi_{i=1}^{n}\Psi(x_i,t)$ in the Stratonovitch representation, for which regular calculus applies, and averaging this. One must then note that each $\Psi(x_i,t)$ depends on ξ_t. Using Eq. (1) again one can express each $\Psi(x_i,t)$ in terms of its value at an earlier time $t-\Delta t$, i.e., $\Psi(x_i,t-\Delta t)$, these being independent of ξ_t. Now averaging is straightforward and performing the limit $\Delta t \to 0$ one obtains Eq. (3). To determine the threshold a_n above which the nth moment increases with time, one needs to calculate the eigenvalues λ_n of Eq. (3). Setting

$$\langle \Psi(x_1,t)\Psi(x_2,t)\cdots\Psi(x_n,t)\rangle = e^{\lambda_n t}\Phi(x_1,x_2,\ldots,x_n) \tag{4}$$

one sees that

$$\left[na-\lambda_n+\sum_{i=1}^{n}\frac{\partial^2}{\partial x_i^2}+\frac{\epsilon}{2}\left(\sum_{i=1}^{n} f\big(x_i\big)\right)^2\right]\Phi(x_1,x_2,\ldots,x_n) = 0. \tag{5}$$

The problem has been reduced to calculating the eigenvalues of an n-particle time-independent Schrödinger equation. When $\epsilon=0$ this equation has solutions of the form $\Phi(x_1,x_2,\ldots,x_n)\approx\exp(\Sigma_i k_j x_j)$ with $\lambda_n=na-\Sigma_i k_i^2$. Calling upon the well-known results of perturbation theory, one can write down the eigenvalues of Eq. (5) to second order in ϵ (see any standard quantum mechanics textbook, e.g., [63]). If one defines $F_{m_1,m_2,\ldots,m_n}$ (m_i integers) to be the Fourier components of the periodic function $\frac{1}{2}[\Sigma_i{}^n f(x_i)]^2$, i.e.,

$$\frac{1}{2}\left(\sum_{i=1}^{n} f(x_i)\right)^2=\sum_{\{m_i\}} e^{iQ(m_1x_1+m_2x_2+\cdots+m_nx_n)}F_{m_1,m_2,\ldots,m_n} \tag{6}$$

one obtains that for the mode $\{k_i\}$

$$\lambda_n=na-\sum_i k_i^2+\epsilon F_{0,0,\ldots,0} + \epsilon^2\sum_{\substack{\{m_i\}\\ \text{not all } m_i=0}}\frac{|F_{m_1,m_2,\ldots,m_n}|^2}{\sum_i k_i^2-\sum_i (k_i+m_iQ)^2}. \tag{7}$$

However, one can write $F_{m_1,m_2,\ldots,m_n}$ in terms of the Fourier components of $f(x)$ and $f^2(x)$. If one further defines

$$\frac{1}{\sqrt{2}}f(x)=\sum_m e^{iQmx}G_m \quad\text{with}\quad \frac{1}{2}f^2(x)=\sum_m e^{iQmx}H_m \tag{8}$$

it is apparent that for all $m_i=0$,

$$F_{0,0,\ldots,0}=nH_0+n(n-1)G_0^2;$$

for $m_i\neq 0$ and all other $m_j=0$,

$$F_{0,0,\ldots,m_i,0,\ldots,0}=H_{m_i}+(n-1)G_{m_i}G_0;$$

for $m_i\neq 0$, $m_j\neq 0$, and all other $m_k=0$,

$$F_{0,0,\ldots,m_i,\ldots,m_j,\ldots,0}=G_{m_i}G_{m_j};$$

and for all other possibilities,

$$F_{m_1,\ldots,m_n}=0.$$

Hence Eq. (7) can be rewritten as

$$\begin{aligned}\lambda_n={}&na-\sum_i k_i^2+\epsilon n H_0+\epsilon n(n-1)G_0^2\\ &+\epsilon^2\sum_{i=1}^{n}\sum_{\substack{m\\ m\neq 0}}\frac{|H_m+(n-1)G_mG_0|^2}{k_i^2-(k_i+mQ)^2}\\ &+\epsilon^2\sum_{i=1}^{n}\sum_{\substack{j=1\\ i\neq j}}^{n}\sum_{\substack{m_1,m_2\\ m_1,m_2\neq 0}}\\ &\times\frac{|G_{m_1}G_{m_2}|^2}{(k_i^2+k_j^2)-(k_i+m_iQ)^2-(k_j+m_jQ)^2}.\end{aligned} \tag{9}$$

For example, taking $f(x)=\cos Qx$ (for which $G_{\pm 1}=1/\sqrt{8}$, $H_0=1/4$, and $H_{\pm 2}=1/8$, with all other $G,H=0$) one finds that

$$\lambda_n=na-\sum_i k_i^2+\frac{n\epsilon}{4}+\epsilon^2\left[\sum_{i=1}^{n}\frac{1}{128(Q^2-k_i^2)} + \frac{1}{16}\sum_{i>j}\left\{\frac{1}{Q^2-(k_i+k_j)^2}+\frac{1}{Q^2-(k_i-k_j)^2}\right\}\right]. \tag{10}$$

To determine the moment thresholds, a_n, one needs the maximal eigenvalues, which occur at zero wave vector, i.e., $\{k_i=0\}$. These are then given by

$$a_n=-\epsilon H_0-\epsilon(n-1)G_0^2+\frac{\epsilon^2}{Q^2}\sum_{m\neq 0}\frac{|H_m+(n-1)G_mG_0|^2}{m^2} + \frac{\epsilon^2(n-1)}{Q^2}\sum_{\substack{m_1,m_2\\ m_1,m_2\neq 0}}\frac{|G_{m_1}G_{m_2}|}{m_1^2+m_2^2} \tag{11}$$

for the general case, and

$$a_n=-\frac{\epsilon}{4}-\frac{\epsilon^2}{128Q^2}(8n-7) \tag{12}$$

for the special case $f(x)=\cos Qx$. This may be compared with the case of spatially constant noise with the same square average $f(x)=1/\sqrt{2}$, where one obtains from Eq. (11) $a_n=-\epsilon n/2$.

Notice that the dependence on the modulation wave vector Q appears only at second order in the noise strength ϵ, and that, for the case of sinusoidal modulation, it is simply of the form Q^{-2}. The first term for the threshold is actually what one obtains when one replaces $[\Sigma_i^n f(x_i)]^2$ by its mean value over space in Eq. (3). Our results (11) and (12) are not valid when $Q=0$, as then the nonzero Fourier components no longer form a discrete set and the nondegenerate perturbation theory is no longer valid. When $Q=0$ the first order term in Eq. (11) is already incorrect as one should obtain that a_n is proportional to n, as calculated above.

As we have been able to calculate all integer moments to second order in ϵ, we can in principle reconstruct the probability distribution for the field itself. One can check that a probability distribution of the form proposed in [29–31] reproduces exactly the moments as calculated here. There the ansatz is made that the logarithm of the norm of the field has a Gaussian distribution. This means that for the critical mode $k_i=0$ the moments increase with time as $e^{\lambda_n t}$, where λ_n contains terms proportional only to n or to n^2 (see Appendix A), which is precisely what one obtains from Eq. (10). The true threshold for the probability distribution of the field is, in this case, given by setting $\lambda_1=\lambda_2/4$ or by setting $n=0$ in the formula for a_n. Hence one concludes that for the general case, the true threshold occurs at

$$a_{\text{threshold}}=-\epsilon H_0+\epsilon G_0^2+\frac{\epsilon^2}{Q^2}\sum_{m\neq 0}\frac{|H_m-G_mG_0|^2}{m^2}+\frac{\epsilon^2}{Q^2}\sum_{\substack{m_1,m_2\\ m_1,m_2\neq 0}}\frac{|G_{m_1}G_{m_2}|}{m_1^2+m_2^2} \tag{13}$$

and for the sinusoidal modulation at

$$a_{\text{threshold}}=-\frac{\epsilon}{4}+\frac{7\epsilon^2}{128Q^2}. \tag{14}$$

Notice that here and throughout the paper we use the subscript n on a control parameter to denote its threshold value for the nth moment and reserve the subscript ''threshold'' to denote values pertaining to the threshold for the probability distribution. Hence, one sees that whereas a spatially constant, temporally fluctuating control parameter produces no shift in the threshold from the deterministic case ($a_n\propto n$, $a_0=a_{\text{threshold}}=0$), a spatially deterministically modulated and temporally fluctuating control parameter is sufficient to change the threshold. It is not necessary to have both the spatial and temporal fluctuations considered in [29,30] to produce a threshold shift.

III. THE SWIFT-HOHENBERG EQUATION

We now turn our attention to the slightly more complicated case of the SHE in one dimension with a similar noise term. This equation has the feature that the threshold for the symmetry breaking instability occurs at nonzero wave vector, which is indeed the case for the experimentally realizable EHC systems. The equation to be studied has the form

$$\frac{\partial}{\partial t}\Psi(x,t)=\left[a-\left(b^2+\frac{\partial^2}{\partial x^2}\right)^2\right]\Psi(x,t)+\sqrt{\epsilon}\cos(Qx)\Psi(x,t)\xi_t+(\text{nonlinear terms}) \tag{15}$$

in the Stratonovitch interpretation. For clarity of presentation we shall, from now on, limit ourselves to the case of a simple sinusoidal spatial modulation of the noise. The following results could be generalized to other periodic functions by expanding the periodic function as a Fourier series.

To determine the threshold we need again only consider the linearized SPDE. Consider briefly the deterministic SHE, Eq. (15), with $\epsilon=0$. This possesses plane wave solutions e^{ikx} just as the deterministic GLE did. They have eigenvalues $a-(b^2-k^2)^2$. The critical mode which becomes unstable first therefore has $k^2=b^2$. As for the GLE the deterministic threshold occurs at $a=0$.

As for the GLE we shall first transform to the Ito representation and then obtain equations for the moments of the stochastic field $\Psi(x,t)$ using Ito calculus. These can be written as

$$\frac{\partial}{\partial t}\langle\Psi(x_1,t)\Psi(x_2,t)\cdots\Psi(x_n,t)\rangle=\left[na-\sum_i^n\left(b^2+\frac{\partial^2}{\partial x_i^2}\right)^2+\frac{\epsilon}{2}\left(\sum_i^n\cos Qx_i\right)^2\right]\times\langle\Psi(x_1,t)\Psi(x_2,t)\cdots\Psi(x_n,t)\rangle. \tag{16}$$

In the absence of noise this equation has solutions of the form $\langle\Psi(x_1,t)\Psi(x_2,t)\cdots\Psi(x_n,t)\rangle\approx\exp(\lambda_n t+i\Sigma_j k_j x_j)$ with $\lambda_n=na-\Sigma_j(b^2-k_j^2)^2$, and so maximizes eigenvalues λ_n when $k_j^2=b^2$ for all j. Our aim is now to determine the eigenvalues λ_n to second order in the noise strength ϵ. To do this we assume that for $\epsilon\neq 0$ the moments take the form

$$\langle\Psi(x_1,t)\Psi(x_2,t)\cdots\Psi(x_n,t)\rangle=C_n\exp\left(\lambda_{nt}+i\sum_j k_jx_j\right)\Phi_n(x_1,x_2,\ldots,x_n), \tag{17}$$

where $\Phi_n(x_1,x_2,\ldots,x_n)$ will be periodic with period $2\pi/Q$ by Floquet's theorem, and C_n are constants. As we only wish to evaluate the thresholds for the moments, a_n, we will only need the maximal eigenvalues and so shall immediately set $k_j^2=b^2$ for all j. We now proceed by expanding the function Φ_n and the eigenvalue λ_n in powers of ϵ, i.e.,

$$\Phi_n(x_1,x_2,\ldots,x_n)=1+\epsilon\phi_n^{(1)}(x_1,x_2,\ldots,x_n)+\epsilon^2\phi_n^{(2)}(x_1,x_2,\ldots,x_n)+\cdots,$$

$$\lambda_n=na+\epsilon\lambda_n^{(1)}+\epsilon^2\lambda_n^{(2)}+\cdots.$$

Substituting the above expansions into the equation for the nth moment, Eq. (16), one sees that at order ϵ,

$$\sum_j\left(-\frac{\partial^4}{\partial x_j^4}-4bi\frac{\partial^3}{\partial x_j^3}+4b^2\frac{\partial^2}{\partial x_j^2}\right)\phi_n^{(1)}=\lambda_n^{(1)}-\frac{1}{2}\left(\sum_j\cos Qx_j\right)^2=\lambda_n^{(1)}-\frac{n}{4}-\frac{1}{4}\sum_j\cos 2Qx_j-\sum_{j>k}\cos Qx_j\cos Qx_k. \tag{18}$$

The function $\phi_n^{(1)}$ must be periodic with period $2\pi/Q$ and hence the constant term on the right-hand side of Eq. (18) must vanish. This immediately gives us that $\lambda_n^{(1)}=n/4$. It is again interesting to note that this is independent of the modulation wave vector Q and is precisely the answer one obtains by replacing the spatial sinusoidal modulations by their average values (i.e., $\cos Qx_i\to 0$, $\cos^2 Qx_i\to 1/2$) in Eq. (16). Solving Eq. (18) one finds that

$$\phi_n^{(1)}=\sum_{j=1}^{n}\ (c_1\cos 2Qx_j+c_2\sin 2Qx_j)$$
$$+\sum_{j>k}[c_3\cos Qx_j\cos Qx_k+c_4\sin Qx_j\sin Qx_k$$
$$+c_5(\sin Qx_j\cos Qx_k+\cos Qx_j\sin Qx_k)],$$
$$Q^2\neq b^2,4b^2, \quad (19)$$

where c_i $(i=1,\ldots,5)$ depend on b and Q.

The separation of periodic behaviors inherent in our ansatz (16) only works when $Q^2\neq p^2b^2$, p integer, so one must exclude these resonances from the range of validity of our calculation.

To determine the second order term, $\lambda_n^{(2)}$, one needs to expand Eq. (16) to second order in ϵ. One then obtains the partial differential equation

$$\sum_j\left(-\frac{\partial^4}{\partial x_j^4}-4bi\frac{\partial^3}{\partial x_j^3}+4b^2\frac{\partial^2}{\partial x_j^2}\right)\phi_n^{(2)}$$
$$=\lambda_n^{(2)}+\lambda_n^{(1)}\phi_n^{(1)}-\frac{1}{2}\phi_n^{(1)}\left(\sum_i\ \cos Qx_i\right)^2. \quad (20)$$

The constant term on the right-hand side of Eq. (20) must again vanish as $\phi_n^{(2)}$ is also periodic and this condition gives one the value of $\lambda_n^{(2)}$. Substituting in the expression for $\phi_n^{(1)}$ from Eq. (19) one concludes that

$$\lambda_n^{(2)}=\frac{nc_1}{8}+\frac{n(n-1)}{16}c_3$$
$$=\frac{n(b^2+Q^2)}{512Q^2(b^2-Q^2)^2}+\frac{n(n-1)(16b^4+Q^4)}{32Q^2(4b^2-Q^2)^2(4b^2+Q^2)},$$
$$Q^2\neq 0,b^2,4b^2. \quad (21)$$

Notice that $\lambda_n^{(2)}$ diverges at the excluded points $Q^2=b^2$ and $4b^2$. At higher order in ϵ such resonances should occur when $Q^2=p^2b^2$ (integer p). A similar structure of divergences would appear for the GLE in Sec. II, when perturbing around a nonzero wave-vector mode, see Eqs. (9) and (10), if one had a nonzero cutoff, for example. As before, our solution is not valid when $Q=0$, as we have assumed a modulation of finite period.

Summarizing our results, one sees that for a sinusoidal noise modulation in the SHE one obtains an eigenvalue for the nth moment,

$$\lambda_n=na+\frac{n\epsilon}{4}+\epsilon^2\left[\frac{n(b^2+Q^2)}{512Q^2(b^2-Q^2)^2}\right.$$
$$\left.+\frac{n(n-1)(16b^4+Q^4)}{32Q^2(4b^2-Q^2)^2(4b^2+Q^2)}\right],\quad Q^2\neq p^2b^2. \quad (22)$$

Again it is apparent that λ_n contains only terms proportional to n and n^2 and so the ansatz of [29–31] is again exact and one concludes that the true threshold for the probability distribution is given by the moment threshold a_n with $n=0$, so that

$$a_{\text{threshold}}=-\frac{\epsilon}{4}+\epsilon^2\left[\frac{(16b^4+Q^4)}{32Q^2(4b^2-Q^2)^2(4b^2+Q^2)}\right.$$
$$\left.-\frac{(b^2+Q^2)}{512Q^2(b^2-Q^2)^2}\right],\quad Q^2\neq p^2b^2. \quad (23)$$

The threshold for sinusoidal modulated noise in the SHE is therefore reduced from the deterministic threshold $a=0$. The shift is identical to that in the GLE, to first order in the noise strength. At second order in the noise strength the shift for the SHE has a more complicated dependence on wave vector than that for the GLE, possessing resonances when the modulation wave vector Q approaches a multiple of the critical wave vector of the system, b. Clearly, for spatially constant noise the situation is the same as in the GLE.

We note here that it is possible to obtain the results of Sec. II using an analogous expansion scheme to that employed here. This simple scheme of separation of the periodicities inherent in the system and then expanding eigenvalues and eigenvectors in powers of the noise strength will be used again in the following section to study the coupled equations obtained for liquid crystals subject to electric fields.

IV. ONE-DIMENSIONAL EHC

We now turn our attention to the most complicated case we shall consider in this paper, the one-dimensional theory of electrohydrodynamic convection in a thin slab of nematic liquid crystal with the director aligned parallel to the slab, say in the x direction. This system has been under intensive experimental study for many years and its near-threshold behavior is understood quite well (for recent reviews on EHC see [61]). The 1D theory gives the simplest set of equations that one can derive directly from the basic equations of hydrodynamics and Maxwell's equations to describe EHC [59,60,58]. In the deterministic case, i.e., for an applied dc or low-frequency ac voltage, it is now mainly of historical interest, but for (spatially constant) stochastic excitation it has been used until recently [16–20].

In this model the spatial degrees of freedom perpendicular to the slab are rather brutally eliminated and, if one is interested only in the most important case where convection rolls run perpendicular to the undistorted nematic director (''normal roll''), one is left with two coupled SPDEs in a single spatial coordinate for the two stochastic fields $q(x,t)$, spatial charge density, and $\psi(x,t)$, the spatial derivative of the angle made by the director with the equilibrium state. For details of the derivation of the equations we refer the reader, e.g., to [58,19]. We adopt here the notation of [19,20]. The price paid for the drastic simplification is that the threshold for the onset of the symmetry breaking instability in the deterministic equations is shifted from the nonzero wave vector and nonzero electric field of the full theory and experiment to zero wave vector and electric field. To attempt to alleviate this unwanted feature, one may artificially restore the nonzero threshold by introducing a cutoff wave vector of appropriate magnitude. This is analogous to working around a nonzero wave vector mode in the GLE, although one knows the critical mode to have zero wave vector. In addition, we have introduced an extra relaxation time $1/T_\psi$ in our coupled equations as an alternative method of restoring a nonzero electric field threshold ($1/T_\psi$ can of course be set equal to zero). Making the one additional simplification that there is no anisotropy in the permittivity of the system, so that the dielectric tensor is isotropic, one obtains the following coupled SPDEs:

$$\frac{\partial}{\partial t}\psi(x,t)=-\frac{1}{T_\psi}\psi(x,t)+K\frac{\partial^2}{\partial x^2}\psi(x,t)-a[E+\sqrt{\epsilon}\cos(Qx)\xi_t]q(x,t), \tag{24}$$

$$\frac{\partial}{\partial t}q(x,t)=-\frac{1}{T_q}q(x,t)-\sigma[E+\sqrt{\epsilon}\cos(Qx)\xi_t]\psi(x,t). \tag{25}$$

The electric field applied to the system is taken to be a constant, E, plus a spatially modulated temporal fluctuation, controlled in magnitude by the parameter ϵ. Again we use a simple sinusoidal modulation with wave vector Q. The parameters K, T_q, a, and σ depend upon material properties of the liquid crystal system and are defined in Appendix B. The assumption of isotropy in the permittivity means that the noise occurs only off-diagonally in the coupled equations. If one allows anisotropy in the permittivity one also needs to consider the square of the control parameter, E. As the square of white noise is ill-defined, it is necessary to introduce a finite correlation time for the temporal noise to allow a controlled calculation, but we do not consider this case here. The SPDEs (24) and (25) are derived from Maxwell's equations and the basic equations of hydrodynamics and are hence to be interpreted in the Stratonovitch sense. Before performing any manipulations with these it is therefore easier to transform to the Ito representation. The equivalent Ito SPDEs can be written concisely using a matrix notation,

$$\frac{\partial}{\partial t}\overline{V}=\bar{\bar{A}}\overline{V}+\bar{\bar{B}}\overline{V}\xi_t, \tag{26}$$

where the vector $\overline{V}=(\psi,q)^\dagger\equiv(v_1,v_2)^\dagger$ and $\bar{\bar{A}}$ and $\bar{\bar{B}}$ are the matrices

$$\bar{\bar{A}}=\begin{pmatrix} -\frac{1}{T_\psi}+K\frac{\partial^2}{\partial x^2}+\frac{a\sigma\epsilon}{2}\cos^2(Qx) & -aE \\ -\sigma E & -\frac{1}{T_q}+\frac{a\sigma E\epsilon}{2}\cos^2(Qx) \end{pmatrix}, \tag{27}$$

$$\bar{\bar{B}}=\begin{pmatrix} 0 & -a\sqrt{\epsilon}\cos Qx \\ -\sigma\sqrt{\epsilon}\cos Qx & 0 \end{pmatrix}. \tag{28}$$

As one now has two fields $\psi(x,t)$ and $q(x,t)$ it is no longer possible to write down a simple explicit equation for the nth moment, as we did for the GLE and SHE. To determine the threshold electric field for the nth moment one now has to evaluate the eigenvalues of a $2^n\times 2^n$ matrix. Here we shall restrict ourselves to the calculation of the thresholds for the first two moments only.

To calculate the first moment threshold one simply has to average Eq. (26) to obtain $\partial\langle\overline{V}\rangle/\partial t=\bar{\bar{A}}\langle\overline{V}\rangle$. One now has to determine the eigenvalues of this equation. It is clear from Eq. (26) that, in the absence of noise, this has solutions of the form $\langle v_j\rangle\propto\exp(\lambda_1 t+ikx)$, and one finds that when $\epsilon=0$

$$\lambda_1=-\frac{1}{2}\left(\frac{1}{T_\psi}+\frac{1}{T_q}+k^2K\right)\pm\frac{1}{2}\sqrt{\left(\frac{1}{T_q}-\frac{1}{T_\psi}-k^2K\right)^2+4a\sigma E^2}. \tag{29}$$

Notice that one obtains the maximal eigenvalue by setting $k=0$ and taking the positive sign, but as discussed above we choose to keep a nonzero cutoff wave vector $k_{\min}$. We shall now employ the same scheme as that outlined in Sec. III to determine the threshold electric field for the first moment, E_1, to second order in ϵ. A Floquet-type ansatz $\langle v_j\rangle=C_j\exp(\lambda_1 t+ikx)\phi_1(x)$ is introduced, where ϕ_1 is again periodic with period $2\pi/Q$. ϕ_1 and λ_1 are then expanded to second order in ϵ,

$$\phi_1=1+\epsilon\phi_1^{(1)}+\epsilon^2\phi_1^{(2)}+\cdots,$$

$$\lambda_1=\lambda_1^{(0)}+\epsilon\lambda_1^{(1)}+\epsilon^2\lambda_1^{(2)}+\cdots, \tag{30}$$

where

$$\lambda_1^{(0)}=-\frac{1}{2}\left(\frac{1}{T_\psi}+\frac{1}{T_q}+k_{\min}^2K\right)+\frac{1}{2}\sqrt{\left(\frac{1}{T_q}-\frac{1}{T_\psi}-k_{\min}^2K\right)^2+4a\sigma E^2}.$$

As for the SHE, one obtains ordinary differential equations for $\phi_1^{(1)}$ and $\phi_1^{(2)}$, which are

$$\left[K\frac{\partial^2}{\partial x^2}+2ik_{\min}K\frac{\partial}{\partial x}\right]\phi_1^{(1)} = \left(1+\frac{a\sigma E^2}{\Omega^2}\right)\left(\lambda_1^{(1)}-\frac{a\sigma}{2}\cos^2 Qx\right), \tag{31}$$

$$\left[K\frac{\partial^2}{\partial x^2}+2ik_{\min}K\frac{\partial}{\partial x}\right]\phi_1^{(2)} = \left(1+\frac{a\sigma E^2}{\Omega^2}\right)\lambda_1^{(2)}+\left(1+\frac{a\sigma E^2}{\Omega^2}\right)\times\left(\lambda_1^{(1)}-\frac{a\sigma}{2}\cos^2 Qx\right)\phi_1^{(1)} - \frac{a\sigma E^2}{\Omega^3}\left(\lambda_1^{(1)}-\frac{a\sigma}{2}\cos^2 Qx\right)^2, \tag{32}$$

where $\Omega=[\lambda_1^{(0)}+(1/T_q)]$.

$\lambda_1^{(1)}$ and $\lambda_1^{(2)}$ are again determined by demanding periodic solutions for $\phi_1^{(1)}$ and $\phi_1^{(2)}$, respectively. From Eq. (31) one sees that $\lambda_1^{(1)}=a\sigma/4$, and then setting this value and the solution of Eq. (31), $\phi_1^{(1)}$, in Eq. (32) one concludes that

$$\lambda_1^{(2)}=\frac{a^3\sigma^3E^2}{32(\Omega^2+a\sigma E^2)\Omega}-\frac{a^2\sigma^2}{128K(k_{\min}^2-Q^2)}\times\left(1+\frac{a\sigma E^2}{\Omega^2}\right),\quad k_{\min}^2\neq Q^2. \tag{33}$$

Summarizing, one sees that

$$\lambda_1=\lambda_1^{(0)}+\epsilon\frac{a\sigma}{4}+\epsilon^2\left[\theta_1+\frac{\theta_2}{K(k_{\min}^2-Q^2)}\right],\quad k_{\min}^2\neq Q^2, \tag{34}$$

where $\theta_{1,2}$ are independent of the modulation wave vector Q. As in the previous two examples, the first correction to λ_1 takes a trivial form and is exactly that obtained by solving Eq. (26) with $\cos^2(Qx)$ replaced by its average value 1/2 and $\cos Qx$ replaced by 0. This correction is once again incorrect when $Q=0$, as our method breaks down (see below for the spatially constant case). At second order we now obtain a contribution independent of the imposed modulation wave vector Q and a term which diverges as Q approaches $k_{\min}$. At higher order in ϵ, resonances at $Q^2=p^2k_{\min}^2$ (p integer) will again emerge and as explained in the preceding setions we have to exclude these points from our range of validity. Inverting the formula in Eq. (34) when $\lambda_1=0$ gives us the threshold electric field for the first moment, E_1:

$$a\sigma E_1^2=\frac{k_{\min}^2K}{T_q}+\frac{1}{T_\psi T_q}-\frac{a\sigma}{4}\left(k_{\min}^2K+\frac{1}{T_q}+\frac{1}{T_\psi}\right)\epsilon + \frac{a^2\sigma^2}{32T_q}\left(\frac{2}{T_q}-k_{\min}^2K-\frac{1}{T_\psi}\right)\epsilon^2+\frac{a^2\sigma^2T_q}{128K(k_{\min}^2-Q^2)} \times\left(k_{\min}^2K+\frac{1}{T_q}+\frac{1}{T_\psi}\right)^2\epsilon^2. \tag{35}$$

Note that the $O(\epsilon)$ correction of the shift does not depend upon Q. It is interesting to note that in spite of the complexity introduced by having two coupled equations, much of the structure of the results obtained in Secs. II and III (e.g., simple form for the first order shift in the threshold and relatively simple dependence of the second order shift on the modulation wave vector Q, with divergences occurring as Q approaches a multiple of the critical wave vector) remains.

To calculate the threshold for the second moments, we obtain a 4×4 matrix equation for the second moments using Ito calculus. This has the form

$$\frac{\partial}{\partial t}\overline{U}(x_1,x_2,t)=\bar{\bar{M}}\,\overline{U}(x_1,x_2,t), \tag{36}$$

where $\overline{U}^\dagger=[\langle\psi(x_1,t)\psi(x_2,t)\rangle,\langle\psi(x_1,t)q(x_2,t)\rangle,\langle q(x_1,t)\psi(x_2,t)\rangle,\langle q(x_1,t)q(x_2,t)\rangle]=(u_1,u_2,u_3,u_4)$, and

$$\bar{\bar{M}}=\begin{pmatrix} A_{11}(x_1)+A_{11}(x_2) & A_{12} & A_{12} & B_{12}(x_1)B_{12}(x_2)\\ A_{21} & A_{11}(x_1)+A_{22}(x_2) & B_{12}(x_1)B_{21}(x_2) & A_{12}\\ A_{21} & B_{12}(x_2)B_{21}(x_1) & A_{11}(x_2)+A_{22}(x_1) & A_{12}\\ B_{21}(x_1)B_{21}(x_2) & A_{21} & A_{21} & A_{22}(x_1)+A_{22}(x_2)\end{pmatrix}, \tag{37}$$

where A_{ij} and B_{ij} are the elements of the matrices $\bar{\bar{A}}$ and $\bar{\bar{B}}$ defined in Eqs. (27) and (28). One now acts analogously to the first moment calculations, introducing the ansatz $u_j(x_1,x_2,t)=D_j\exp(\lambda_2t+ik_1x_1+ik_2x_2)\phi_2(x_1,x_2)$, where D_j are constants and ϕ_2 is periodic with period $2\pi/Q$, and then expanding in powers of ϵ: $\lambda_2=2\lambda_1^{(0)}+\epsilon\lambda_2^{(1)}+\epsilon^2\lambda_2^{(2)}+\cdots$ and $\phi_2=1+\epsilon\phi_2^{(1)}+\epsilon^2\phi_2^{(2)}+\cdots$.

We carried out this procedure using the mathematical package MAPLE. One then obtains partial differential equations to solve for $\phi_2^{(1)}$ and $\phi_2^{(2)}$ and demanding that these solutions be periodic leads one to $\lambda_2^{(1)}$ and $\lambda_2^{(2)}$. $\lambda_2^{(1)}$ again has a simple form $\lambda_2^{(1)}=a\sigma/2=2\lambda_1^{(1)}$. This leads one to speculate that $\lambda_n^{(1)}=na\sigma/4$ analogous in form to the GLE and SHE. $\lambda_2^{(2)}$ has a very complicated form and we shall show here only the dependence on the modulation wave vector Q,

$$\lambda_2^{(2)}=\frac{1}{(k_{\min}^2-Q^2)^2}\left[d_1Q^4+d_2Q^2+d_3+\frac{d_4}{Q^2}\right]+\frac{1}{(k_{\min}^2-Q^2)}[d_5Q^2+d_6]+\frac{1}{(4k_{\min}^2-Q^2)^2}\left[d_7Q^4+d_8Q^2+d_9+\frac{d_{10}}{Q^2}\right]+\frac{1}{(4k_{\min}^2-Q^2)}[d_{11}Q^2+d_{12}]+\frac{d_{13}}{Q^4}+\frac{d_{14}}{Q^2}+d_{15}, \tag{38}$$

where d_j are independent of Q. Although the expression for $\lambda_2^{(2)}$ is now very complicated, the characteristic resonances at $Q^2=0,k_{\min}^2$ and $4k_{\min}^2$ are still present. To find the electric field threshold for the second moment, E_2, one needs to invert the formula $2\lambda_1^{(0)}+\epsilon\lambda_2^{(1)}+\epsilon^2\lambda_2^{(2)}=0$, but we shall not do this here as the interesting features are already evident in the expressions for $\lambda_2^{(1),(2)}$. It would be interesting to attempt to determine λ_3 to second order in ϵ to try to verify if the ansatz of Refs. [29–31] is still valid for this more complicated case. Although we have not attempted this cumbersome task as we obtain the same thresholds for the first two moments to first order in ϵ, it seems reasonable to assume that this is in fact the true threshold for the probability distribution to first order in ϵ.

For explicit comparison we have also calculated the case of spatially constant noise. Replacing in Eqs. (24), (25) the function $\cos Qx$ by $1/\sqrt{2}$ we find $\lambda_1^{(1)}=a\sigma/4$ (unchanged), but now $\lambda_1^{(2)}=0$, and

$$\lambda_2^{(1)}=\frac{a\sigma}{2}+\frac{2a^2\sigma^2E^2}{\Gamma^2+4a\sigma E^2}, \tag{39}$$

$$\lambda_2^{(2)}=\frac{1}{8}\frac{a^2\sigma^2\Gamma^4}{[\Gamma^2+4a\sigma E^2]^{5/2}}+\frac{2a^3\sigma^3E^2\Gamma^2}{[\Gamma^2+4a\sigma E^2]^{5/2}}, \tag{40}$$

with $\Gamma=1/T_q-1/T_\psi-k_{\min}^2K$.

Instead of having $\lambda_2^{(1)}=2\lambda_1^{(1)}$ as in the previous cases, we now see that $2\lambda_1^{(1)}\leqslant\lambda_2^{(1)}\leqslant 4\lambda_1^{(1)}$, the limiting case $\lambda_2^{(1)}=2\lambda_1^{(1)}$ holding when $E=0$, i.e., the driving of the system is purely stochastic. For this case of spatially constant noise we have also calculated the third moment exponents to second order in epsilon (the answers $\lambda_3^{(1),(2)}$ are available on request from the authors). This allows us to verify if the ansatz of [29] still holds. We find that the ansatz always holds to first order in ϵ, but is only true to second order in ϵ for the case of purely stochastic driving. To first order in ϵ we find that the threshold electric field takes the value

$$a\sigma E_{\text{threshold}}^2=\frac{1}{T_qT_\psi}+\frac{k_{\min}^2K}{T_q}-\frac{a\sigma}{4}\Lambda\epsilon+\frac{a\sigma}{T_q\Lambda}\left(\frac{1}{T_\psi}+k_{\min}^2K\right)\epsilon, \tag{41}$$

where $\Lambda=(1/T_\psi+k_{\min}^2K+1/T_q)$. It is interesting to compare this threshold with that for the case of spatially modulated noise which we have conjectured to be the expression (35) to first order in ϵ. One sees that unless one of the effective correlation times $1/T_q$ or $k_{\min}^2K+1/T_\psi$ vanishes, the threshold for spatially constant noise is always greater than that for spatially modulated noise. The first order shift of the threshold takes a maximal value of zero when the two effective correlation times are equal, i.e., $1/T_q=k_{\min}^2K+1/T_\psi$. In principle the results for spatially constant noise (at least as far as the moments are concerned) are included in the theory of Refs. [16–20]. We have not been able to make direct connection, since in that work the case of isotropic permittivity appears as a complicated (singular) limit, and moreover, the dichotomous nonwhite noise used there becomes Gaussian and white only in another singular limit.

V. CONCLUSION

We have considered SPDEs in one dimension with multiplicative noise in which the noise is a product of white noise in time and a deterministic periodic function in space. The results are contrasted with those for spatially constant noise. For the GLE and SHE we have determined the thresholds for the integer moments of the field to second order in the noise strength away from resonances which occur when the periodic function modulating the noise has a wave vector given by a multiple of the critical threshold wave vector. These thresholds, a_n, were seen to take the form required by a recent ansatz introduced by Becker and Kramer. Having obtained all the integer moments one is able to determine the true threshold for the probability distribution itself, which in these cases is then given by setting $n=0$ in the general formula for a_n. One sees that, in contrast to the case in which one has spatially constant noise, where there is no shift in the threshold with increasing noise strength, one obtains for spatially modulated noise a threshold which decreases with the noise strength and is dependent upon the modulating function and its wave vector. The case of spatially modulated white noise therefore contains the essential ingredients for a threshold shift which had already been observed and calculated for the GLE and SHE with noise which is the product of white noise in time and white noise in space [21,29–31,22,23].

Further, we have considered the case of two coupled SPDEs obtained in the one-dimensional theory of electrohydrodynamics in liquid crystals. Here we were only able to calculate the first two moments of the coupled fields. The eigenvalues giving the leading long time behavior of the first two moments took similar overall forms to those obtained for the GLE and SHE. For the case of spatially constant noise we could calculate the first three moments and hence conclude that here the ansatz of [29–31] holds only to first order in the noise strength. As we obtained identical thresholds for the first two moments for spatially modulated noise to first order in ϵ, we assume that this is also the threshold for the probability distribution itself and that the ansatz holds here to first order. It would be interesting to determine the threshold for the third moment to check that this is correct and see if the ansatz also fails at second order for the spatially modulated case. This would be a possible, if extremely cumbersome, calculation using the scheme outlined in this paper. Comparison of the thresholds for spatially modulated and spatially constant noise to first order in the noise strength showed that the threshold for spatially constant noise always exceeds that for spatially modulated noise, except in the limit where one of the inverse relaxation times vanishes. To first order in ϵ one can in fact interpolate for the case of spatially constant noise between the threshold shift for spatially modulated noise and zero threshold shift by varying the relaxation times. It would be interesting to try to detect these threshold shifts in experimentally realizable systems. For this purpose spatially structured electrodes could be used in EHC experiments. Although a direct quantitative comparison is questionable using this one-dimensional theory with a deterministic dc electric field (usually an ac field is applied), perhaps the more qualitative features (e.g., existence of a threshold shift and possibly its dependence on the external wave vector Q) could be detected. To enable better comparison with experiment it would obviously be desirable to extend these calculations to systems of equations giving a better description of the experimental system, for example allowing a dielectric anisotropy, and possibly using the two-dimensional theory for liquid crystals in an electric field [19,20,61]. This, however, seems a rather hard task due to the increased complexity of the coupled equations for the two-dimensional theory, with the large number of spatial derivatives contained therein. We would like to mention that there are nematic liquid crystal materials with vanishing, or nearly vanishing, dielelectric anisotropy. In fact one such material has recently been introduced in EHC experiments [64].

From the present work, as well as the previous one [29–31], it follows that a typical feature displayed by systems subjected to spatially distributed multiplicative noise is that the behavior of the system is to leading nontrivial order in the noise strength determined already by the first moment of the distribution function. This feature has been stressed recently in other works where a type of mean-field approximation was used as an analytical starting point [22,54,55,24]. Additive noise of (in principle) arbitrary strength was included. In the presence of not too weak (external) additive noise a reentrant behavior with increasing multiplicative noise strength was found in the GLE, which presumably occurs also in the SHE [22,24], i.e., strong multiplicative noise could destroy again the order that it supported initially. As far as we see these interesting effects are out of reach of our approach. However, finding a suitable experimental system may also be very difficult. One can have a noise-induced phase transition also in models that have no transition at all in the absence of noise [54,55].

Recent work by Grinstein *et al.* [65] has considered simple SPDEs with spatially and temporally stochastic multiplicative noise from the point of view of theory of phase transitions (renormalization group). In particular, it was suggested that for the GLE there exists a critical dimension $d_c=2$ above which there is no threshold shift for noise strength below some nonzero value. It would be interesting to see if such features (if they really occur) also persist when the spatial stochasticity is replaced by spatial periodicity, but are absent for spatial homogeneity, as is the case for the threshold shift in the single SDE.

Finally we mention that a large system is not needed to obtain the effects discussed in this paper. In fact the following system of two coupled ordinary linear SDEs shows the essential features [29,30]

$$\frac{d}{dt}\psi_1(t)=[a+\sqrt{\epsilon}\xi_1(t)]\psi_1(t)-\psi_1(t)+\psi_2(t), \quad (42)$$

$$\frac{d}{dt}\psi_2(t)=[a+\sqrt{\epsilon}\xi_2(t)]\psi_2(t)-\psi_2(t)+\psi_1(t). \quad (43)$$

The coupling of the equations is of the (discrete) diffusion type. If the noise processes $\xi_1(t)$ and $\xi_2(t)$ are taken to be equal, we are back to the one-component model with $a_{\text{threshold}}=0$. If $\xi_1(t)$ and $\xi_2(t)$ are uncorrelated Gaussian white noise processes one finds (see [29,30])

$$a_{\text{threshold}}=-\frac{\epsilon}{4}+\frac{\epsilon^2}{32}+O(\epsilon^4), \quad (44)$$

which is analogous to the spatially distributed case.

Actually stochastic driving is not needed. One can replace the functions $\xi_i(t)$ by periodic functions and find similar effects. The stability exponent λ (which is now a Floquet exponent) is equal to a if the functions are equal. If one chooses the amplitudes to be different, one easily obtains a result analogous to Eq. (44). Generally the shift results from an interaction between the spatially inhomogeneous driving and the spatial coupling, which tends to flatten out inhomogeneities. Such systematic forces resulting from oscillatory and spatially varying excitation (not necessarily parametric) are known in plasma physics under the name of ''ponderomotive forces'' (see, e.g., [66]). Similar effects occur in nematic liquid crystals subjected to an oscillatory flow with a nonlinear flow field (e.g., Poiseuille flow) [67].

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APPENDIX A

In this appendix we demonstrate that if one employs the ansatz for the probability distribution of the stochastic field above threshold introduced in Refs. [29–31], the moments of the field have eigenvalues with terms proportional only to n and n^2.

The ansatz assumes that in the long-time limit the logarithm of the norm of the stochastic field $\ln|\Psi(x,t)|\equiv\rho(x,t)$ has a Gaussian distribution with mean mt and width σt, where we have adopted the notation of [30], i.e.,

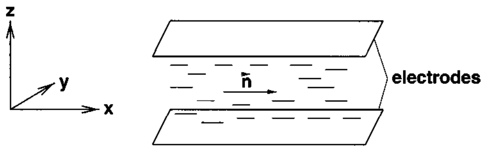

FIG. 1. Geometry of the experimental situation used to derive the 1D equations of EHC.

$$P\{\rho(x,t)\}=\frac{1}{\sqrt{2\pi\sigma t}}\exp\left[-\frac{(\rho-mt)^2}{2\sigma t}\right]. \qquad \text{(A1)}$$

The moments of the field can then be seen to be given for integer n by

$$\langle\Psi^{2n}(x,t)\rangle=\langle e^{2n\rho}\rangle,$$

$$\langle\Psi^{2n+1}(x,t)\rangle=\text{sgn}(\Psi(x,t))\langle e^{(2n+1)\rho}\rangle, \qquad \text{(A2)}$$

so that

$$\langle\Psi^n(x,t)\rangle\sim\frac{1}{\sqrt{2\pi\sigma t}}\int\mathcal{D}\rho(x,t)\exp\left[n\rho-\frac{(\rho-mt)^2}{2\sigma t}\right]$$

$$=\exp\left[\frac{n^2\sigma t}{2}+nmt\right]. \qquad \text{(A3)}$$

The moments hence have eigenvalues λ_n such that

$$\lambda_n=\frac{n^2\sigma}{2}+nm, \qquad \text{(A4)}$$

i.e., the eigenvalues can only have a term proportional to n and a term proportional to n^2.

In this framework the true threshold occurs when $\langle\rho(t)\rangle\to$const as $t\to\infty$, i.e., when $m\to 0$ for large time or equivalently

$$\lambda_1=\frac{\lambda_2}{4} \text{ or } \left.\frac{\lambda_n}{n}\right|_{n=0}=0. \qquad \text{(A5)}$$

APPENDIX B

The SPDEs (24) and (25) are derived from a consideration of the following experimental situation. One has a nematic liquid crystal layer confined by two flat plate electrodes, lying in the xy plane with the director of the nematic liquid crystal along a unit vector $\vec{n}$, see Fig. 1 (note that for a director $\vec{n}=-\vec{n}$).

We adopt the notation used in [19]. For this system we can write down the viscous angular momentum per unit volume as

$$\vec{\Gamma}_{\text{vis}}=-\vec{n}\times[(\alpha_3-\alpha_2)\vec{N}+(\alpha_3+\alpha_2)\hat{A}\cdot\vec{n}], \qquad \text{(B1)}$$

where the α's are the Leslie viscosity coefficients. $\vec{N}$, defined as

$$\vec{N}=\frac{\partial\vec{n}}{\partial t}+\vec{v}\cdot\text{grad}\vec{n}-\frac{1}{2}\text{rot}\vec{v}\times\vec{n}, \qquad \text{(B2)}$$

describes the motion of the director relative to the hydrodynamic velocity of the flowing liquid $\vec{v}$ and $\hat{A}$ is the symmetric part of the velocity tensor

$$\hat{A}=\frac{1}{2}(\partial_j v_i+\partial_i v_j). \qquad \text{(B3)}$$

The stress tensor has both elastic and viscous parts. The viscous part t'_{ij} can be written in terms of six constants α_1, α_2, α_3, α_4, α_5, and α_6 as

$$t'_{ij}=\alpha_1 n_i n_j\sum_{k,l} n_l A_{lk} n_k+\alpha_2 n_i N_j+\alpha_3 N_i n_j+\alpha_4 A_{ij}$$

$$+\frac{1}{2}(\alpha_5+\alpha_6-\alpha_2-\alpha_3)n_i\sum_k n_k A_{kj}$$

$$+\frac{1}{2}(\alpha_5+\alpha_6+\alpha_2+\alpha_3)n_j\sum_k A_{ik}n_k. \qquad \text{(B4)}$$

Further, our nematic has the conductivity $\sigma_{ij}=\sigma_\perp\delta_{ij}+\sigma_a n_i n_j$, an isotropic permittivity $\epsilon_{ij}=\epsilon_\perp\delta_{ij}$, and an elastic bending constant K_{33}.

It is helpful to define effective shear viscosities:

$$\eta_1\equiv\frac{1}{2}\left[\alpha_4-\alpha_2-\alpha_3+\frac{1}{2}(\alpha_5+\alpha_6+\alpha_3-\alpha_2)\right], \qquad \text{(B5)}$$

$$\eta\equiv\alpha_3-\alpha_2-\frac{\alpha_2^2}{\eta_1}. \qquad \text{(B6)}$$

One can now express the coefficients of Eqs. (24) and (25) in terms of the material parameters of the nematic as follows:

$$K=K_{33}/\eta, \qquad \text{(B7)}$$

$$a=-\alpha_2/\eta\eta_1, \qquad \text{(B8)}$$

$$1/T_q=4\pi(\sigma_\perp+\sigma_a)/\epsilon_\perp, \qquad \text{(B9)}$$

$$\sigma=\sigma_a. \qquad \text{(B10)}$$

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